

NEW ERROR TERM FOR THE FOURTH MOMENT OF AUTOMORPHIC L -FUNCTIONS

OLGA BALKANOVA AND DMITRY FROLENKOV

ABSTRACT. We improve the error term in the asymptotic formula for the twisted fourth moment of automorphic L -functions of prime level and weight two proved by Kowalski, Michel and Vanderkam. As a consequence, we obtain a new subconvexity bound in the level aspect and improve the lower bound on proportion of simultaneous non-vanishing.

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1. INTRODUCTION

The fourth moment of automorphic L -functions has been studied in [3, 7] using the large sieve inequality and δ -symbol method. As an application Duke, Friendlander and Iwaniec proved the subconvexity bound in the level aspect. Another consequence – simultaneous non-vanishing – was derived by Kowalski, Michel and Vanderkam.

In this paper, we optimize several estimates of [7] and compute the explicit dependence of error terms on the smallest positive eigenvalue for the Hecke congruence subgroup. This allows us to improve the results of [3, 7] by applying the Kim-Sarnak bound.

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We borrow some notations of [3, 7]. Consider the family $H_2^*(q)$ of primitive newforms of prime level q and weight 2. Every $f \in H_2^*(q)$ has a Fourier expansion

$$(1.1) \quad f(z) = \sum_{n \geq 1} \lambda_f(n) n^{1/2} e(nz).$$

The associated L -function is defined by

$$(1.2) \quad L(f, s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}, \quad \Re s > 1.$$

The completed L -function

$$(1.3) \quad \Lambda(f, s) = \left(\frac{\sqrt{q}}{2\pi} \right)^s \Gamma\left(s + \frac{1}{2}\right) L(f, s)$$

can be analytically continued on the whole complex plane. It satisfies the functional equation

$$(1.4) \quad \Lambda(f, s) = \epsilon_f \Lambda(f, 1 - s), \quad \epsilon_f = \pm 1.$$

We introduce the natural and harmonic averages

$$(1.5) \quad \sum_{f \in H_2^*(q)}^n \alpha_f := \sum_{f \in H_2^*(q)} \frac{\alpha_f}{|H_2^*(q)|}, \quad \sum_{f \in H_2^*(q)}^h \alpha_f := \sum_{f \in H_2^*(q)} \frac{\alpha_f}{4\pi \langle f, f \rangle_q},$$

where $\langle f, f \rangle_q$ is the Petersson inner product on the space of level q holomorphic modular forms.

The goal of the present paper is to improve the error term in the asymptotic formula for the twisted fourth moment

$$(1.6) \quad M(l) = \sum_{f \in H_2^*(q)}^h \lambda_f(l) |L(f, 1/2 + \mu)|^4, \quad \mu \in i\mathbb{R}.$$

Our main result is the following.

Theorem 1.1. *Let q be a prime and $l < q$. There exists some $B > 0$ such that for any $\epsilon > 0$*

$$(1.7) \quad M(l) = M^D(l) + M^{OD}(l) + M^{OOD}(l) + O_\epsilon \left(q^\epsilon (1 + |\mu|)^B \left(l^{\frac{5-6\theta}{8-8\theta}} q^{-\frac{1-2\theta}{8-8\theta}} + l^{\frac{17}{8}} q^{-\frac{1}{4}} + l^{\frac{5-4\theta}{8-8\theta}} q^{-\frac{1}{8-8\theta}} \right) \right),$$

where $M^D(l)$, $M^{OD}(l)$ and $M^{OOD}(l)$ are the main terms defined by equations (17), (31) – (32) and (34) of [7].

Here

$$(1.8) \quad \theta := \sqrt{\max(0, 1/4 - \lambda_1)}$$

and $\lambda_1 = \lambda_1(q)$ is the smallest positive eigenvalue for the Hecke congruence subgroup $\Gamma_0(q)$. Currently the best known estimate on λ_1 is due to Kim and Sarnak [8]. Accordingly, we can take

$$\theta = 7/64.$$

Corollary 1.2. *Let q be a prime. For all $\epsilon > 0$*

$$(1.9) \quad M(1) = P(\log q) + O_\epsilon(q^{-25/228+\epsilon}),$$

where P is a polynomial of degree 6 and the leading coefficient is $1/60\pi^2$.

This improves corollary 1.3 of [7], where asymptotic formula (1.9) was established with the error $O_\epsilon(q^{-1/12+\epsilon})$.

Note that for weight $k > 2$ the remainder term in (1.9) can be majorated by $O_{\epsilon,k}(q^{-1/4+\epsilon})$. This was proved in [1] for the case of prime power level $q = p^\nu$, $\nu > 2$.

Another consequence of theorem 1.1 is a new subconvexity bound in the level aspect.

Corollary 1.3. *For all $\epsilon > 0$*

$$(1.10) \quad L(f, 1/2 + \mu) \ll_{\epsilon, \mu} q^{1/4-\delta},$$

where $\delta = \frac{2\theta-1}{16(8\theta-7)}$.

Taking $\theta = 7/64$, we obtain

$$\delta = \frac{25}{3136} = \frac{1}{125.44}.$$

The previously known result with $\delta = 1/192$ was established by Duke, Friedlander and Iwaniec [3].

2. SELBERG'S EIGENVALUE CONJECTURE

Let Γ be a congruence subgroup of modular group. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$ be the eigenvalues of the automorphic Laplacian on $L^2(\Gamma \backslash \mathbb{H})$ induced from the Laplace operator

$$(2.1) \quad \Delta_L = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The eigenvalue $0 < \lambda < 1/4$ is called an exceptional eigenvalue.

Conjecture 2.1. (Selberg, [12]) *The Laplacian for a congruence subgroup has no exceptional eigenvalues, i.e. $\lambda_1 \geq 1/4$.*

Below we provide several results related to conjecture 2.1:

- 1965 Selberg [12]: $\lambda_1 \geq 3/16$;
- 1978 Jacquet and Gelbart [4]: $\lambda_1 > 3/16$;
- 1995 Luo, Rudnick, Sarnak [10]: $\lambda_1 > 171/784$;
- 1996 Iwaniec [6]: $\lambda_1 > 10/49$;
- 2002 Kim, Shahidi [9]: $\lambda_1 \geq 66/289$;
- 2003 Kim, Sarnak [8]: $\lambda_1 \geq 975/4096$.

Using the bound of Kim-Sarnak and equation (1.8), we find

$$(2.2) \quad \theta = \sqrt{\max(0, 1/4 - \lambda_1)} = 7/64.$$

3. LARGE SIEVE INEQUALITY

Let $S(m, n; c)$ be the classical Kloosterman sum.

Theorem 3.1. (theorem 9 of [2] and lemma 9 of [11]) *Let r, s and d be positive pairwise coprime integers with r and s square-free. Let C, M, N be positive real numbers and g be real-valued infinitely differentiable function with support in $[M, 2M] \times [N, 2N] \times [C, 2C]$ such that*

$$(3.1) \quad \left| \frac{\partial^{(j+k+l)}}{\partial m^{(j)} \partial n^{(k)} \partial c^{(l)}} g(m, n, c) \right| \leq M^{-j} N^{-k} C^{-l} \text{ for } 0 \leq j, k, l \leq 2.$$

Let

$$X_d := \frac{\sqrt{dMN}}{sC\sqrt{r}}.$$

Then for any $\epsilon > 0$ and complex sequences $\mathbf{a} = \{a_m\}$, $\mathbf{b} = \{b_n\}$ one has

$$(3.2) \quad \sum_m a_m \sum_n b_n \sum_{\substack{c \\ (c,r)=1}}^c g(m, n, c) S(dm\bar{r}, \pm n; sc) \ll_\epsilon C^\epsilon d^\theta sC\sqrt{r} \frac{(1 + X_d^{-1})^{2\theta}}{1 + X_d} \left(1 + X_d + \sqrt{\frac{M}{rs}}\right) \left(1 + X_d + \sqrt{\frac{N}{rs}}\right) \\ \times \left(\sum_{M < m \leq 2M} |a_m|^2\right)^{1/2} \left(\sum_{N < n \leq 2N} |b_n|^2\right)^{1/2},$$

where θ is defined by equation (1.8).

4. ERROR TERMS

In this section, we consider the terms that give the largest contribution to the error in [7]. Our goal is to optimize the estimates of these terms and compute the exact dependence of the error on parameter θ .

First, we improve bound (21) of [7].

Note that the function $F_{M,N}(m, n)$ defined on page 108 of [7] is compactly supported on $[M/2, 3M] \times [N/2, 3N]$ and

$$(4.1) \quad F_{M,N}(x, y) \ll (1 + |\mu|)^B (MN)^{-1/2}.$$

Lemma 4.1. *Assume that for any $\epsilon > 0$ one has $M, N \ll q^{1+\epsilon}$. Then for any $C > \sqrt{lMN}$*

$$(4.2) \quad \sum_{de=l} \frac{1}{d^{1/2}} \sum_{ab=d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{\substack{c \geq C \\ q|c}} \frac{1}{c^2} T_{M,N}(c) \ll_{\epsilon} (1 + |\mu|)^B (Cq)^{\epsilon} l^{1/2} \left(\frac{\sqrt{MN}}{C} \right)^{1-2\theta}.$$

Proof. We split $[C, \infty)$ into dyadic intervals and take $c \in [C, 2C]$. By equation (18) of [7] we have

$$\begin{aligned} \sum_{q|c} \frac{1}{c^2} T_{M,N}(c) &= \sum_{n,m} \sum_{q|c} \tau(m) \tau(n) \frac{1}{c} S(m, aen; c) \\ &\times J_1 \left(\frac{4\pi \sqrt{aemn}}{c} \right) F_{M,N}(m, n) = \frac{1}{q} \sum_{n,m} \tau(m) \tau(n) \\ &\times \sum_{c_1} \frac{1}{c_1} S(m, aen, c_1 q) J_1 \left(\frac{4\pi \sqrt{aemn}}{c_1 q} \right) F_{M,N}(m, n). \end{aligned}$$

Here $m \in [M/2, 3M]$, $n \in [N/2, 3N]$ and $c_1 \in [C_1, 2C_1]$ with $C_1 := C/q$. Let

$$Y := \sqrt{MNC_1} \left(\frac{\sqrt{aeMN}}{C} \right)^{-1}.$$

As a test function we choose

$$g(m, n, c_1) := \frac{Y}{c_1} F_{M,N}(m, n) J_1 \left(\frac{4\pi \sqrt{aemn}}{c_1 q} \right).$$

It satisfies condition (3.1), and theorem 3.1 can be applied with $d = ae$, $r = 1$ and $s = q$. Hence

$$\begin{aligned} \sum_{de=l} \frac{1}{d^{1/2}} \sum_{ab=d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{\substack{q|c \\ c \geq C}} \frac{1}{c^2} T_{M,N}(c) &\ll_{\epsilon} (1 + |\mu|)^B (Cq)^{\epsilon} l^{1/2} \left(\frac{\sqrt{MN}}{C} \right)^{1-2\theta}. \end{aligned}$$

□

The optimal value of C can be chosen by making equal the estimate (4.2) and the first summand of equation (26) of [7], namely

$$(4.3) \quad l^{1/2} \left(\frac{\sqrt{MN}}{C} \right)^{1-2\theta} = l^{3/4} \frac{N^{1/4}}{M^{1/2}} \frac{C}{q}.$$

This gives

$$(4.4) \quad C = l^{-\frac{1}{8-8\theta}} \min \left(q^{\frac{1}{2-2\theta}} \sqrt{MN}^{\frac{1-4\theta}{8-8\theta}}, q^{\frac{9-8\theta}{8-8\theta}} \right).$$

After performing the dyadic summation over M and N , we find that for any $l < q^{\frac{1}{5-4\theta}}$ the error term in lemma 4.1 is bounded by

$$(4.5) \quad O_\epsilon \left(q^\epsilon (1 + |\mu|)^B l^{\frac{5-6\theta}{8-8\theta}} q^{-\frac{1-2\theta}{8-8\theta}} \right).$$

Now we consider two other error terms that depend on C . These are the errors resulting from extension of summation over $c > C$. See section 3.5 (pages 111 – 112) of [7].

Let

$$(4.6) \quad \eta_C(c) := \begin{cases} 1 & c \leq C \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 4.2. *Let C be defined by equation (4.4). For any $\epsilon > 0$*

$$(4.7) \quad \sum_{M, N \ll q^{1+\epsilon}} \sum_{de=l} \frac{1}{d^{1/2}} \sum_{ab=d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OD} \ll_\epsilon (1 + |\mu|)^B q^\epsilon l^{(5-4\theta)/(8-8\theta)} q^{-1/(8-8\theta)}.$$

Proof. Consider

$$\begin{aligned} \sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OD} &= -2\pi \sum_n \tau(aen) \tau(n) \\ &\times \int_0^\infty Y_0(4\pi\sqrt{aent}) J_1(4\pi\sqrt{aent}) \sum_{\substack{q|c \\ c>C}} \phi(c) F_{M,N}(c^2t, n) dt. \end{aligned}$$

Since $C^2t < c^2t \leq 2M$, the sum over c can be estimated as follows

$$\sum_{\substack{q|c \\ c>C}} \phi(c) F_{M,N}(c^2t, n) \ll \frac{1}{\sqrt{MN}} \frac{M}{qt}.$$

Next we apply $Y_0(x) \ll \log x$ and $J_1(x) \ll x$. Then

$$\begin{aligned} \sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OD} &\ll_{\epsilon} \\ (1 + |\mu|)^B q^{\epsilon} \frac{N}{\sqrt{MN}} \int_0^{2M/C^2} t^{\epsilon} \frac{M}{qt} (aeNt)^{1/2} dt &\ll_{\epsilon} \\ (1 + |\mu|)^B q^{\epsilon} (ae)^{1/2} \frac{MN}{qC}. \end{aligned}$$

Finally, using (4.4), we obtain

$$\begin{aligned} \sum_{M, N \ll q^{1+\epsilon}} \sum_{de=l} \frac{1}{d^{1/2}} \sum_{ab=d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OD} &\ll_{\epsilon} \\ (1 + |\mu|)^B q^{\epsilon} l^{(5-4\theta)/(8-8\theta)} q^{-1/(8-8\theta)}. \end{aligned}$$

□

Lemma 4.3. *Let C be defined by equation (4.4). For any $\epsilon > 0$*

$$\begin{aligned} (4.8) \quad \sum_{M, N \ll q^{1+\epsilon}} \sum_{de=l} \frac{1}{d^{1/2}} \sum_{ab=d} \frac{\mu(a)}{a^{1/2}} \tau(b) \sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OOD} &\ll_{\epsilon} \\ (1 + |\mu|)^B q^{\epsilon} l^{(5-4\theta)/(8-8\theta)} q^{-1/(8-8\theta)}. \end{aligned}$$

Proof. According to [7] page 111 we have

$$\sum_{q|c} (1 - \eta_C(c)) c^{-2} T^{OOD} \ll_{\epsilon} (1 + |\mu|)^B q^{\epsilon} (ae)^{1/2} \frac{MN}{qC}.$$

Equation (4.4) yields the assertion.

□

To sum up, the largest error terms in theorem 1.1 come from lemmas 4.1, 4.2, 4.3 and equation (26) of [7]. In particular, the error term $O_{\epsilon}((1 + |\mu|)^B l^{17/8} q^{-1/4+\epsilon})$ is given by the second summand in (26) of [7].

5. AMPLIFICATION AND SUBCONVEXITY

Contribution of the main terms M^D , M^{OD} , M^{OOD} in [7] is bounded by

$$(5.1) \quad O_{\epsilon}((1 + |\mu|)^B q^{\epsilon} l^{-1/2}).$$

According to theorem 1.1, for $l < q^{\frac{1}{12-11\theta}}$ we have

$$(5.2) \quad \sum_{f \in H_2^*(q)} \frac{1}{4\pi \langle f, f \rangle_q} \lambda_f(l) |L(f, 1/2 + \mu)|^4 \ll_{\epsilon, \mu} q^\epsilon \left(l^{-1/2} + l^{\frac{5-6\theta}{8-8\theta}} q^{-\frac{1-2\theta}{8-8\theta}} \right).$$

Let

$$(5.3) \quad \Lambda_f(\mathbf{c}) := \sum_{\substack{l \leq L \\ (l, q) = 1}} c_l \lambda_f(l)$$

be an amplifier. Then

$$(5.4) \quad \sum_{f \in H_2^*(q)} \frac{1}{4\pi \langle f, f \rangle_q} \Lambda_f^2(\mathbf{c}) |L(f, 1/2 + \mu)|^4 \ll_{\epsilon, \mu} q^\epsilon \left(\|\mathbf{c}\|_2^2 + L^{\frac{5-6\theta}{4-4\theta}} q^{-\frac{1-2\theta}{8-8\theta}} \|\mathbf{c}\|_1^2 \right),$$

where $\|\mathbf{c}\|_p$ denotes l_p -norm.

We choose coefficients c_l as in [3], making $\Lambda_f(\mathbf{c})$ large for a particular form $f \in H_2^*(q)$, namely

$$(5.5) \quad c_l = \begin{cases} \lambda_f(l) & \text{if } l \text{ is prime } \leq L^{1/2} \\ -1 & \text{if } l \text{ is a square of a prime } \leq L^{1/2} \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$(5.6) \quad \Lambda_f(\mathbf{c}) = \sum_{\substack{l \text{ prime } \leq L^{1/2} \\ (l, q) = 1}} (\lambda_f(l)^2 - \lambda_f(l^2)).$$

Note that $\lambda_f(l)^2 - \lambda_f(l^2) = 1$ for prime l such that $(l, q) = 1$. Therefore,

$$(5.7) \quad \Lambda_f(\mathbf{c}) \sim 2L^{1/2}(\log L)^{-1}.$$

By Deligne's bound

$$(5.8) \quad \|\mathbf{c}\|_2^2 \leq 5\Lambda_f(\mathbf{c}) \text{ and } \|\mathbf{c}\|_1 \leq 3\Lambda_f(\mathbf{c}).$$

The results of [5] imply that

$$(5.9) \quad \frac{1}{4\pi \langle f, f \rangle_q} \ll \frac{\log q}{q}.$$

Taking $L = q^{\frac{2\theta-1}{2(8\theta-7)}}$ in (5.3) and applying (5.7), (5.8), (5.9), we have

$$(5.10) \quad L(f, 1/2 + \mu) \ll_{\epsilon, \mu} q^{1/4-\delta}$$

with $\delta = \frac{2\theta-1}{16(8\theta-7)}$.

6. MOLLIFICATION AND SIMULTANEOUS NON-VANISHING

We follow section 5.2 of [7]. In order to determine the largest admissible length of mollifier Δ , we sum the error terms in theorem 1.1 against $l^{-1/2+\epsilon}$ for $l < q^{2\Delta}$. This gives

$$(6.1) \quad q^{-\frac{1-2\theta}{8-8\theta}} q^{2\Delta(\frac{1-2\theta}{8-8\theta}+1)+\epsilon} + q^{-1/4} q^{21\Delta/4+\epsilon} + q^{-\frac{1}{8-8\theta}} q^{2\Delta(\frac{1}{8-8\theta}+1)+\epsilon}.$$

Therefore, the error term is negligible for any $\Delta < \frac{1-2\theta}{2(9-10\theta)}$.

In order to change the harmonic mean into the natural average as defined by (1.5), we apply results of section 5. Accordingly, condition (82) of [7] is satisfied for any $\Delta < \frac{1-2\theta}{4(7-8\theta)}$.

Theorem 6.1. *Let $M(f)$ be the mollifier defined by equation (63) of [7] with $P(x) = x^3$. Let $F(\Delta)$ be defined by equation (5) of [7].*

For all $0 < \Delta_1 < \frac{1-2\theta}{2(9-10\theta)}$ we have

$$(6.2) \quad \sum_{f \in H_2^*(q)}^h L(f, 1/2)^4 M(f)^4 = (1 + o(1)) F(\Delta_1) \left(\frac{\zeta(2)}{\log q} \right)^4.$$

For all $0 < \Delta_2 < \frac{1-2\theta}{4(7-8\theta)}$ we have

$$(6.3) \quad \sum_{f \in H_2^*(q)}^n L(f, 1/2)^4 M(f)^4 = (1 + o(1)) F(\Delta_2) \left(\frac{\zeta(2)}{\log q} \right)^4.$$

Taking $\theta = 7/64$, we find that $\Delta_1 < \frac{25}{566} = \frac{1}{22.64}$ and $\Delta_2 < \frac{25}{784} = \frac{1}{31.36}$. This improves $\Delta_1 < \frac{1}{30}$ and $\Delta_2 < \frac{1}{48}$ proved in [7].

In particular, extension of admissible length of mollifier Δ_2 gives a better lower bound on the proportion of simultaneous non-vanishing

$$(6.4) \quad \sum_{\substack{f \in H_2^*(q) \\ L(f, 1/2) L(f \otimes \chi, 1/2) \neq 0}} 1,$$

where χ is a fixed primitive character of conductor D such that $(D, q) = 1$. See Proposition 7.2 of [7] for the exact formulas.

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INSTITUTE FOR APPLIED MATHEMATICS OF RUSSIAN ACADEMY OF SCIENCES,
Khabarovsk, Russia

E-mail address: olgabalkanova@gmail.com

STEKLOV MATHEMATICAL INSTITUTE OF RUSSIAN ACADEMY OF SCIENCES,
NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, MOSCOW,
Russia

E-mail address: frolenkov@mi.ras.ru